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Cycles in hamiltonian graphs of prescribed maximum degree

Antoni Marczyk, Mariusz Woźniak

Faculty of Applied Mathematics A G H, Al. Mickiewicza 30, 30-059 Kraków, Poland

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Abstract

Let G be a hamiltonian graph G of order n and maximum degree Δ , and let $C(G)$ denote the set of cycle lengths occurring in G . It is easy to see that $|C(G)| \geq \Delta - 1$. In this paper, we prove that if $\Delta > n/2$, then $|C(G)| \geq (n + \Delta - 3)/2$. We also show that for every $\Delta \geq 2$ there is a graph G of order $n \geq 2\Delta$ such that $|C(G)| = \Delta - 1$, and the lower bound in case $\Delta > n/2$ is best possible.

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1. Introduction

We consider only finite, undirected and simple graphs and we use Bondy and Murty's book [1] for terminology. In particular, for a graph G , we denote by $V = V(G)$ its vertex set and by $E = E(G)$ its set of edges. By C_p we mean a p -cycle of G , i.e. a cycle of length p . The vertices of a graph G of order n will be denoted by integers $1, 2, \dots, n$. The edges of a complementary graph of a graph G are referred to as *red edges*. We denote by $C(G)$ the set of integers p , $3 \leq p \leq n$, such that G contains a cycle of length p .

The description of the set $S(n, \Delta)$ of cycle lengths occurring in every hamiltonian graph of order n and maximum degree Δ is given in [3,4]. Clearly, $S(n, \Delta) = \bigcap C(G)$, where the intersection is taken over all hamiltonian graphs of order n and maximum degree Δ . In particular, it is shown in [4] that a hamiltonian graph of order n

E-mail addresses: marczyk@uci.agh.edu.pl (A. Marczyk), mwozniak@uci.agh.edu.pl (M. Woźniak).

and $\Delta > n/2$ contains a cycle C_p for every integer p belonging to the union

$$\bigcup_{s=1}^n \left(\frac{n-1-\Delta}{s} + 2, \frac{\Delta}{s} + 2 \right).$$

This result was shown [4] to be best possible.

There are several results (see [2–7]) on the set of cycle lengths in a hamiltonian graph with given degree sum of two vertices.

The purpose of the present paper is the study of the number $|C(G)|$ of cycles lengths rather than the structure of $C(G)$. We give a lower bound of this number in dependence of the maximum degree Δ and the order of the graph. In particular we shall observe a “jump” of this bound in the neighborhood of the value $\Delta = n/2$. More precisely we shall prove the following theorem.

Theorem 1. *Let G be a hamiltonian graph of order n and maximum degree Δ . If $\Delta \leq n/2$, then $|C(G)| \geq \Delta - 1$. Moreover, for every $\Delta \geq 2$ there exist a graph G for which this bound is attained.*

If $\Delta > n/2$, then $|C(G)| \geq n/2 + \Delta/2 - 3/2$. This bound is best possible.

The rest of the paper is organized as follows. The easy case of small values of Δ is considered below. Section 2 contains some lemmas. The last section is devoted to the second inequality of Theorem 1.

We shall use the following notation. The symbol G stands for a hamiltonian graph of order n with vertex set $[1, n] = \{1, 2, 3, \dots, n-1, n\}$ and edge set E . By $C = (1, 2, 3, \dots, n-1, n, 1)$ we denote a hamiltonian cycle of G . The degree of the vertex 1 is Δ , the maximum degree of G . The set of neighbors of 1 will be denoted by X . Note that with this notation, if $p \in X$ and $2 < p < n$, then $p \in C(G)$.

It is easily checked that for a hamiltonian graph G of maximum degree $\Delta \leq n/2$, $|C(G)| \geq \Delta - 1$. This is, in a sense, best possible in view of the construction below.

Let $k \geq 4$ and $q \geq 0$ be two integers. Define G as follows. The order of G equals $(q+2)(k-2)+2$ and the edge-set of G consists of the hamiltonian cycle $1, 2, 3, \dots, n-1, n, 1$ and of the edges joining 1 with every vertex of the form $k+x(k-2)$, where $0 \leq x \leq q$. It is easy to see that G has maximum degree $\Delta = q+3$ and that the cycles of G may have only $q+1$ lengths of the form $k+x(k-2)$, $0 \leq x \leq q$ and, of course, one cycle of length n . Thus $|C(G)| = q+2 = \Delta - 1$.

2. Some lemmas

For given $A \subset V$ we denote by $f(A)$ the number of neighbors of 1 in A i.e.

$$f(A) = |X \cap A|.$$

We start with some simple observations.

Proposition 2. *If $k \notin C(G)$, then $k \notin X$ and $n - k + 2 \notin X$.*

Proposition 3. *If $k \notin C(G)$ and $a \in X$ and $a + k - 2 < n$, then $a + k - 2 \notin X$.*

Corollary 4. *Let A and B be two disjoint subsets of $[1, n]$ with $B = A + (k - 2)$. If $k \notin C(G)$ then*

$$f(A \cup B) = f(A) + f(B) \leq |A| = \frac{1}{2}(|A| + |B|).$$

Proof. The proof follows from the observation that if $x \in A \cap X$ then, by Proposition 3, the vertex $x + (k - 2)$ belonging to B is not in X . \square

In particular, we shall use the last corollary when A and B are two consecutive segments (i.e. their union is also a segment) containing each $k - 2$ elements. However, in this case we shall need a more general result.

Lemma 5. *Let B_1, B_2, \dots, B_{2t} be $2t$ disjoint, consecutive segments of $[1, n]$, each of length $k - 2$. If $k \notin C(G)$, then*

$$f\left(\bigcup_{i=1}^{2t} B_i\right) \leq \frac{1}{2} \sum_{i=1}^{2t} |B_i|.$$

Proof. Since the number of segments B_i is $2t$, we can divide the segments into t pairs $(B_1, B_2), (B_3, B_4), \dots, (B_{2t-1}, B_{2t})$ and apply Corollary 4 to each pair separately. By adding the obtained inequalities we get the conclusion. \square

Since, for $t > 1$, B_1 and B_{2t} are not consecutive, the value of $f(B_1 \cup B_{2t})$ may be greater than $|B_1|$. However, in this case we have the following estimation.

Lemma 6. *With the same notation as in the previous lemma suppose that $k \notin C(G)$. Then*

$$f(B_1 \cup B_{2t}) \leq |B_1| + \xi,$$

where ξ is defined by

$$f\left(\bigcup_{i=2}^{2t-1} B_i\right) = \frac{1}{2} \sum_{i=2}^{2t-1} |B_i| - \xi.$$

Proof. The assertion of the lemma follows from Corollary 4 in the case $t=1$. Therefore, suppose $t > 1$. Applying the previous lemma to the sequence of segments B_2, \dots, B_{2t-1} we deduce that $\xi \geq 0$. It suffices now to repeat the previous step to the sequence B_1, \dots, B_{2t} . \square

The key lemma is the following:

Lemma 7. *If $|C(G)| < n/2 + \Delta/2 - 3/2$, then there exists an integer p , $p \leq (n+2)/2$, such that*

$$p \notin C(G) \quad \text{and} \quad n - p + 2 \notin C(G).$$

Proof. Let $C(G)^c = [3, n] \setminus C(G)$. We note that if $(n+2)/2 \in C(G)^c$, then $(n+2)/2$ is the desired integer p . Assume that $(n+2)/2 \notin C(G)^c$. If $k \in C(G)^c$, then Proposition 2 states that $k \notin X$ and $n - k + 2 \notin X$. Now if there exist distinct $k, l \in C(G)^c$ such that $\{k, n - k + 2\}$ intersects $\{l, n - l + 2\}$, then $l = n - k + 2 \in C(G)^c$, and k is the desired integer p . So, we suppose the sets $\{k, n - k + 2\}$ with $k \in C(G)^c$ are pairwise disjoint. By Proposition 2, this implies that the number of red edges incident with vertex 1 is at least $2|C(G)^c|$. The degree of vertex 1 is then at most $n - 1 - 2|C(G)^c| < n - 1 - (n - \Delta - 1) = \Delta$, a contradiction. \square

3. Large values of Δ

Suppose, contrary to our claim, that there is a graph G of order n and $\Delta > n/2$ such that $|C(G)| < n/2 + \Delta/2 - 3/2$. Let p , $3 \leq p \leq (n+2)/2$ be an integer satisfying the following property:

$$p \notin C(G) \quad \text{and} \quad n - p + 2 \notin C(G). \quad (*)$$

The existence of p is guaranteed by Lemma 7. If $p < (n+2)/2$ we have $n - 2(p - 2) - 3 \geq 0$ vertices between p and $n - p + 2$ on C . Let t and r be the quotient and the remainder when $n - 2p + 1$ is divided by $2(p - 2)$, i.e.

$$n - 2p + 1 = 2t(p - 2) + r \quad (**)$$

with $0 \leq r < 2(p - 2)$. If $p = (n+2)/2$ we put $t = r = 0$.

Let r_1, r_2 be two integers such that $r_1 + r_2 = r$, $0 \leq r_1 \leq r_2 \leq r_1 + 1$. For $r_1 \geq 1$ we define two segments on C

$$R_1 = [p + 1, p + r_1],$$

$$R_2 = [n - p - r_2 + 2, n - p + 1].$$

In other words R_1 is the segment having r_1 vertices with first vertex $p + 1$ and R_2 is the segment having r_2 vertices with last vertex $n - p + 1$. For $r_1 = 0$ or $r_2 = 0$ the corresponding set R_i is, by definition, empty.

Denote by B the segment $[p + r_1 + 1, n - p - r_2 + 1]$. By the construction, the segment B consists of an even number of segments, each of lengths $(p - 2)$.

We put $V_1 = \{i - (p - 2) : i \in R_1\}$ and $V_2 = \{i + (p - 2) : i \in R_2\}$. Hence $V_1 = [3, r_1 + 2]$ and $V_2 = [n - r_2, n - 1]$. Of course, if R_i is empty, then the set V_i is empty too.

Finally denote by U_1, U_2 the remaining parts of the segments $[3, p]$ and $[n - p + 2, n - 1]$, respectively. In other words $U_1 = [r_1 + 3, p]$ and $U_2 = [n - p + 2, n - r_2 - 1]$. Observe that the segments $U_1 \cup R_1$ and $U_2 \cup R_2$ are both of length $p - 2$ (see Fig. 1). By Lemma 5 we know that $f(B) \leq (1/2)|B|$.

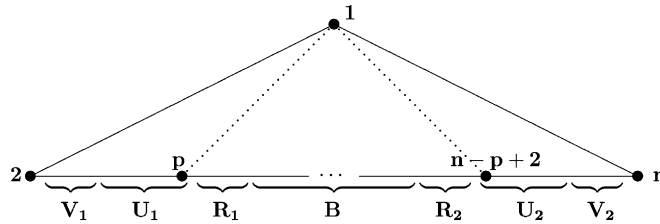


Fig. 1.

Let us put

$$f(B) = (1/2)|B| - \xi. \quad (1)$$

Applying Lemma 6 to the sequence of segments $U_1 \cup R_1, B, R_2 \cup U_2$ and using (1) we get

$$f(U_1 \cup R_1) + f(R_2 \cup U_2) \leq p - 2 + \xi. \quad (2)$$

Applying Corollary 4 to the sets V_1 and R_1 , as well as to the sets V_2 and R_2 , we get

$$f(V_1) + f(R_1) \leq |R_1| = r_1, \quad (3)$$

$$f(V_2) + f(R_2) \leq |R_2| = r_2. \quad (4)$$

Consider the set $V_1 \cup U_1 \cup U_2 \cup V_2$. Suppose that there exists $x \in [3, p-1]$ with $x \in X$. Then $(n-p+x) \notin X$, for otherwise we would have a cycle of length $n-p+2$ defined by $1, x, x+1, \dots, n-p+x, 1$, which contradicts the property (*). By symmetry, we therefore obtain

$$f(V_1) + f(U_1) + f(U_2) + f(V_2) \leq p - 3. \quad (5)$$

Let us put $A = V_1 \cup U_1 \cup R_1 \cup R_2 \cup U_2 \cup V_2$. Observe that $|A| = 2(p-2) + r_1 + r_2$. Moreover, $n = 3 + |A| + |B|$. Adding the inequalities (2), (3), (4) and (5) we get

$$2f(A) \leq p - 2 + \xi + r_1 + r_2 + p - 3.$$

Hence

$$2f(A) \leq |A| + \xi - 1.$$

Thus

$$f(A) \leq \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2}.$$

Using the last inequality, (1) and the fact that the edges $(1, 2)$ and $(1, n)$ are in E we get

$$\Delta = 2 + f(A) + f(B) \leq 2 + \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2} + \frac{|B|}{2} - \xi \leq \frac{|A| + |B| + 3}{2} = \frac{n}{2},$$

a contradiction.

Finally, for given n define a graph G of order n and maximum degree Δ as follows: the edge-set of G consists of the hamiltonian cycle $1, 2, 3, \dots, n-1, n, 1$ and of the edges joining 1 with every vertex x where $(n-\Delta+5)/2 \leq x \leq (n+\Delta-1)/2$ if $n-\Delta$ is odd and $(n-\Delta+4)/2 \leq x \leq (n+\Delta-2)/2$ if $n-\Delta$ is even.

It is easy to see that G has indeed maximum degree Δ and that G have no cycle of length greater than $(n+\Delta-1)/2$ if $n-\Delta$ is odd and $(n+\Delta)/2$ if $n-\Delta$ is even (except for the cycle of length n). Thus $|C(G)| = (n+\Delta-3)/2$ if $n-\Delta$ is odd and $|C(G)| = (n+\Delta-2)/2$ if $n-\Delta$ is even. This finishes the proof of the theorem.

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